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# On Additive Lyapunov Functions and Existence of Neutral Supply Rates in Acyclic LTI Dynamical Networks *

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***Abstract*— In this paper we are concerned with linear time invariant (LTI) systems which admit a Lyapunov function with a specific additive structure. We prove that if a dynamical network, composed as set of LTI systems interconnected over an acyclic graph, admits an additive quadratic Lyapunov function, then the systems are dissipative with respect to a set of interconnection neutral supply rates (we show that this set is necessarily nonempty), where each supply rate from the set is defined on a single interconnection link in the network.**

## I. INTRODUCTION

Dissipativity theory [1] has been one of the major tools in both *i)* robust control theory, where many of the problems can be formulated, solved or interpreted in this framework; *ii)* stability analysis / control synthesis for large scale systems, see e.g. [1] and [5] for classical results and, e.g., [3], [4] for a more recent controller synthesis result. One of the fundamental results states that if interconnected systems are dissipative with suitably defined *interconnection neutral supply rates*, then the overall interconnected system is stable (see e.g. [1], [2]).

In this paper we consider dynamical networks defined as a set of linear time invariant systems interconnected over an arbitrary acyclic graph. We define an additive Lyapunov function as a Lyapunov function which is a sum of "local functions", where each such local function is assigned to one system in the network and depends only on the states of that particular system. Indeed, the above mentioned dissipativity-type results commonly end up with additive Lyapunov functions as the main analysis/synthesis tools, where the local functions are nothing else than the storage functions related to the interconnection neutral supply rates.

While it is well known that existence of neutral supply rates implies existence of an additive Lyapunov function, to the best of our knowledge, the converse statement has not been proven in a sufficiently general case. In this paper we prove such converse result for the case of acyclic dynamical networks of LTI systems and quadratic additive Lyapunov functions. More precisely, we prove that if such dynamical network is stable and admits an additive Lyapunov function, then there necessarily exists a set of suitably defined quadratic interconnection neutral supply rates defined on the interconnection links. We restrict ourself to systems in which the direct feed-through matrices (the "$D$ matrix" in a state-space realization) of systems in the network are zero.

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## II. NOTATION AND PRELIMINARIES

In this section we define the notation and present some notions and results which will be instrumental in the remainder of the paper.

*1) Notation.:* Let $\mathbb{R}$ denote the field of real numbers and let $\mathbb{R}^{m\times n}$ denote $m$ by $n$ matrices with elements in $\mathbb{R}$. The transpose of a matrix $A$ is denoted by $A^\top$. We use $\mathbb{S}^n$ to denote the set of all symmetric matrices of dimension $n\times n$. Ker$A$ and Im$A$ are used to denote the kernel and the image space of $A$, respectively. The operator $\mathrm{col}(\cdot,\ldots,\cdot)$ stacks its operands into a column vector, and $\mathrm{diag}(\cdot,\ldots,\cdot)$ denotes a square matrix with its operands on the main diagonal and zeros elsewhere. The matrix inequalities $A\succ B$ ($A\prec B$) and $A\succeq B$ ($A\preceq B$) mean $A$ and $B$ are symmetric and $A-B$ is positive definite (negative definite) and positive semi-definite (negative semi-definite), respectively. Blocks in matrices that can be inferred by symmetry are sometimes denoted by $\star$ to save space. For a finite set $\Omega$ we use to $|\Omega|$ to denote its cardinality. For a linear time invariant (LTI) system $G$ with a state-space description

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} \tag{1}$$

we will use the notation $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ or $G=(A,B,C,D)$ to denote a state space realization of a transfer function $G(s)$, i.e., $G(s) = C(sI-A)^{-1}B+D$.

*2) Dissipative LTI systems with quadratic supply rates:* Here we briefly recall characterization of dissipative LTI systems in terms of linear matrix inequalities. For more details we refer to, e.g., [1], [2]. We say that an LTI system $G$ given by (1) is strictly dissipative with respect to the quadratic *supply function*

$$s(d,z) = \begin{pmatrix} d \\ z \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} d \\ z \end{pmatrix},$$

where $Q$ and $R$ are symmetric matrices of appropriate dimensions and $S$ is a real matrix, if there exists a quadratic *storage function* $V(x) = x^\top Px$, such that the time derivative of $V(x(t))$ along the system's trajectory satisfies the inequality

$$\dot{V}(x(t)) < s(d(t), z(t))$$

at any time $t$ and for all $\mathrm{col}(x(t), d(t), z(t)) \neq 0$. This dissipativity condition is equivalet to the existence of a symmetric

$P$ such that the following linear matrix inequality (LMI) is feasible

$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^{\top} \left(\begin{array}{cc:cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hdashline 0 & 0 & -Q & -S \\ 0 & 0 & -S^{\top} & -R \end{array}\right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \prec 0.$$

*3) Interconnection neutral supply rates:* Consider two systems $G_1$ and $G_2$ given by

$$G_1 : \begin{cases} \dot{x}_1 = A_1x_1 + B_1v_1 \\ w_1 = C_1x_1 + D_1v_1 \end{cases}, \quad G_2 : \begin{cases} \dot{x}_2 = A_2x_2 + B_2v_2 \\ w_2 = C_2x_2 + D_2v_2 \end{cases} \tag{2}$$

and their interconnection obtained by taking $v_1 = w_2$ and $v_2 = w_1$. Indeed, we assume that the dimensions of the considered signals are compatible so that such interconnection is possible. Suppose that the system $G_1$ is strictly dissipative with respect to supply function $s_1(v_1,w_1)$ and system $G_2$ is strictly dissipative with respect to supply function $s_2(v_2,w_2)$, that is, there exist storage functions $V_1(x_1)$ and $V_2(x_2)$, such that for all systems trajectories the following dissipation inequalities hold

$$\begin{aligned} \dot{V}_1(x_1) &< s_1(v_1,w_1), \quad \text{for} \quad \mathrm{col}(x_1,v_1,w_1) \neq 0 \\ \dot{V}_2(x_2) &< s_2(v_2,w_2) \quad \text{for} \quad \mathrm{col}(x_2,v_2,w_2) \neq 0. \end{aligned} \tag{3}$$

The interconnection is said to be *neutral* with respect to supply rates $s_1, s_2$ if

$$s_1(v_1,w_1) + s_2(v_2,w_2) = 0,$$

for all $v_1$, $w_1$, $v_2$, $w_2$ such that $v_1 = w_2$, $v_2 = w_1$.

**Remark II.1** *The following two conditions imply stability of the interconnected systems:*

1) *The dissipation inequalities* (3) *are satisfied with positive definite storage functions, that is, $V_i(x_i) > 0$ for all $x_i \neq 0$, $i = 1,2$;*
2) *The interconnection is neutral with respect to the supply rates $s_1$ and $s_2$ from* (3).

*Indeed, positive definite function $V(x_1,x_2) := V_1(x_1)+V_2(x_2)$ has negative definite time derivative along the state trajectories of the interconnected system, since the above two conditions imply $\dot{V}(x_1,x_2) = \dot{V}_1(x_1)+\dot{V}_2(x_2) < s_1(v_1,w_1)+s_2(v_2,w_2) = 0$ and therefore $V(x_1,x_2)$ is a Lyapunov function for the interconnected system.*

*4) Non-conservative stability result based on full-block S-procedure:* Consider system interconnection presented in Figure 1 and given by $w = Gv$, $v = \Delta(w)$. Let $G$ be an LTI

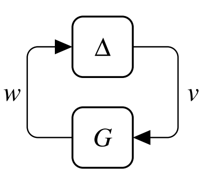


Fig. 1. Uncertain dynamical system represented in a basic feedback interconnection

system and suppose the operator $\Delta$, which belongs to some set $\Lambda$, is a static (memoryless) mapping from $\mathbb{R}^{n_w}$ to $\mathbb{R}^{n_v}$. The following theorem considers the case when $\Lambda$ is a *compact set*, and is a consequence (in particular the *"only if"* part) of the full-block S-procedure. For more details see [6], [7].

**Theorem II.2** *Let $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n_w \times n_v}$ and let $\Lambda$ be a* compact set. *Then the system presented in Figure 1 is (exponentially) stable* if and only if *there exist $P \in \mathbb{S}^n$, $P \succ 0$ and $Q \in \mathbb{S}^{n_v}$, $R \in \mathbb{S}^{n_w}$, $S \in \mathbb{R}^{n_v \times n_w}$ such the following matrix inequalities are satisfied*

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^{\top} \begin{pmatrix} Q & S \\ S^{\top} & R \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad \text{for all} \quad \Delta \in \Lambda,$$

$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^{\top} \left(\begin{array}{cc:cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S \\ 0 & 0 & S^{\top} & R \end{array}\right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \prec 0.$$

## III. DYNAMICAL NETWORKS

In this section we introduce the notion of dynamical networks, as used in this paper, and present a suitable modelling framework for such systems. The section is to a large extent following the modelling framework from [8].

We define a *dynamical network* as a finite set of dynamical systems interconnected via physical or communicational links over some graph. More precisely, we use a directed graph $\Gamma := (\Omega, E)$ in which each vertex $G_i \in \Omega$ is identified with a dynamical system, while a directed edge $(G_i, G_j) \in E$ means that the dynamics of the system $G_i$ influences the dynamics of the system $G_j$, i.e., there is an output signal of $G_i$ that is input to $G_j$.

Let $L = |\Omega|$, i.e., $L$ is the number of vertices (systems) in $\Gamma$. We assume that system indexes range from 1 to $L$, that is, $\Omega = \{G^i\}_{i=1,\ldots,L}$. We will use the following notation for the interconnection signals:

- $w_{ij}$ is the signal associated with the directed edge $(G_i, G_j)$, i.e., $w_{ij}$ is an output from the system $G_i$ and influences dynamics of the system $G_j$. We use $n_{ij}$ to denote the spatial dimension of $w_{ij}$, that is, $w_{ij}(t) \in \mathbb{R}^{n_{ij}}$.
- $v_{ji}$ is the signal associated with the directed edge $(G_i, G_j)$, denotes the input signal to the system $G_j$.

When the interconnections between the system are ideal (e.g., there are no time delays, or dynamical elements in general, in the interconnection links) we have the following interconnection relations

$$v_{ij} = w_{ji} \tag{5}$$

for all edges in $\Gamma$. Note that $v_{ij}(t) \in \mathbb{R}^{n_{ji}}$. With the following abbreviations

$$\begin{aligned} v &:= \underset{i=1,\ldots,L}{\mathrm{col}} \left( \underset{j=1,\ldots,L}{\mathrm{col}} (v_{ij}) \right) \\ w &:= \underset{i=1,\ldots,L}{\mathrm{col}} \left( \underset{j=1,\ldots,L}{\mathrm{col}} (w_{ij}) \right) \\ w_H &:= \underset{i=1,\ldots,L}{\mathrm{col}} \left( \underset{j=1,\ldots,L}{\mathrm{col}} (w_{ji}) \right) \end{aligned}$$

the ideal interconnections (5) are in compact way given by

$$v = w_H = Hw,$$

where $H$ is suitably defined permutation matrix. We will refer to the matrix $H$ as the *interconnection matrix*.

For each $i \in \{1,\ldots,L\}$ we further make the following definitions

$$v_i := \operatorname*{col}_{j=1,\ldots,L}(v_{ij}),$$
$$w_i := \operatorname*{col}_{j=1,\ldots,L}(w_{ij}),$$

that is, the signal $v_i$ collects all interconnection signals which act as an input to the system $G_i$, while the signal $w_i$ collects all outputs of the system $G_i$. We limit our focus to finite dimensional, linear, time-invariant systems, and in that case the system $G_i$ can be represented in a state-space form as follows

$$\begin{pmatrix} \dot{x}_i \\ w_i \end{pmatrix} = \begin{pmatrix} A_i & B_i \\ C_i & 0 \end{pmatrix} \begin{pmatrix} x_i \\ v_i \end{pmatrix},$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state vector. We will use $G_i(s)$ to address the transfer matrix of the $i$-th system, i.e., we have

$$G_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & 0 \end{array}\right].$$

Finally, with the abbreviations $x := \operatorname*{col}_{i=1,\ldots,L}(x_i)$ and

$$A := \operatorname*{diag}_{i=1,\ldots,L}(A_i), \quad B := \operatorname*{diag}_{i=1,\ldots,L}(B_i), \quad C := \operatorname*{diag}_{i=1,\ldots,L}(C_i),$$

the overall interconnected system is given by

$$\begin{pmatrix} \dot{x} \\ w \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}, \quad v = Hw. \tag{7}$$

With $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ the overall interconnected system is presented in Figure 2. Note that $G(s) = \operatorname*{diag}_{i=1,\ldots,L}(G_i(s))$, that is, the system $G$ is a collection of uncoupled systems $G_i$, collected together in a transfer matrix $G$ with block diagonal structure.

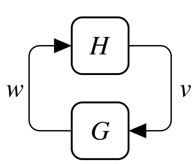


Fig. 2. Spatially distributed system represented as a set of uncoupled dynamical systems ($G$) interconnected via matrix $H$ in feedback loop.

Let $n := \sum_{i=1}^{L} n_i$, $m := \sum_{i=1}^{L}\sum_{j=1}^{L} n_{ij}$, where $n_{ij} = 0$ when $(G^i, G^j) \notin E$. Based on Theorem II.2 we have the following stability result.

**Proposition III.1** *The system* (7) *is stable* if and only if *there exists* $P \in \mathbb{S}^{n\times n}$, $P \succ 0$, *and* $Q \in \mathbb{S}^{m\times m}, S \in \mathbb{R}^{m\times m}, R \in \mathbb{S}^{m\times m}$, *such that*

$$\begin{pmatrix} H \\ I \end{pmatrix}^{\top} \begin{pmatrix} Q & S \\ S^{\top} & R \end{pmatrix} \begin{pmatrix} H \\ I \end{pmatrix} = R + S^{\top}H + H^{\top}S + H^{\top}QH \succeq 0 \tag{8}$$

*and the following matrix inequality is satisfied*

$$\begin{pmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{pmatrix}^{\top} \begin{pmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & Q & S \\ 0 & 0 & S^{\top} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{pmatrix} \prec 0. \tag{9}$$

The conditions of Proposition III.1, when satisfied, imply that $V(x) := x^{\top}Px$ is Lyapunov function for the dynamical network.

## IV. PROBLEM DEFINITION AND OVERVIEW OF THE MAIN RESULTS

### A. Problem definition

We say that the dynamical network given by (7) admits an additive Lyapunov function if there exists a block diagonal matrix $P = \operatorname{diag}(P_1,\ldots,P_L)$ with $P_i \in \mathbb{S}^{n_i}$, such that $P \succ 0$ and $\mathscr{A}^{\top}P + P\mathscr{A} \prec 0$, where $\mathscr{A} = A + BHC$. The term *additive* comes from the fact that the Lyapunov function is then given by

$$V(x) = \underbrace{x_1^{\top}P_1x_1}_{V_1(x_1)} + \ldots \underbrace{x_L^{\top}P_Lx_L}_{V_L(x_L)}, \tag{10}$$

that is, $V(x)$ is a sum of local functions $V_i(\cdot)$, where each $V_i$ is *local* to the system $i$ in a sense that it depends only on the states of that system.

Before formally stating the problem definition, we further make the following assumptions and definitions.

Let $\hat{\Gamma} = (\Omega, \hat{E})$ be an undirected graph defined from the directed interconnection graph $\Gamma = \{\Omega, E\}$ as follows

$$\Big((G_i,G_j) \in E\Big) \text{ or } \Big((G_j,G_i) \in E\Big) \implies (G_i,G_j) \in \hat{E},$$
$$\Big((G_i,G_j) \notin E\Big) \text{ and } \Big((G_i,G_j) \notin E\Big) \implies (G_i,G_j) \notin \hat{E},$$

We make the following assumption.

**Assumption IV.1** *The graph* $\hat{\Gamma}$ *is acyclic.*

Let $N_i$ denote the set of indices of the systems adjacent to the system $G_i$ in $\hat{\Gamma}$, that is, $N_i := \{j \,|\, (G_i,G_j) \in \hat{E}\}$. In connection to the system $G_i$ we define the following set of supply functions, each related to one edge $(G_i,G_j)$, $j \in N_i$:

$$s_{ij}(v_{ij}, w_{ij}) := \begin{pmatrix} v_{ij} \\ w_{ij} \end{pmatrix}^{\top} \Pi_{ij} \begin{pmatrix} v_{ij} \\ w_{ij} \end{pmatrix}, \quad j \in N_i, \tag{11}$$

where $\Pi_{ij}$ is a symmetric real matrix of suitable dimensions.

**Theorem IV.2** *With Assumption IV.1, the following two statements are equivalent:*

i) *The system* (7) *admits an additive quadratic Lyapunov function of the form* (10).
ii) *For each* $i \in \Omega$ *and each* $j \in N_i$ *there exists a symmetric real matrix* $\Pi_{ij}$, *which defines* $s_{ij}$ *as in* (11), *so that*
   a) $\dot{V}_i(x_i) < \sum_{j\in N_i} s_{ij}(v_{ij}, w_{ij})$ *along trajectories* $x_i$, $v_{ij}$, $w_{ij}$ *satisfying* (7);
   b) $s_{ij}(v_{ij}, w_{ij}) + s_{ji}(v_{ji}, w_{ji}) = 0$ *for each* $(i,j)$ *such that* $(G_i,G_j) \in \hat{E}$;

*where* $V_i(x_i) = x_i^\top P_i x_i$.

Note that the inequalities in part ($a$) of the statement ($ii$) mean that the system $G_i$ is strictly dissipative with respect to the supply function $\sum_{j \in N_i} s_{ij}(v_{ij}, w_{ij})$, while the condition in part ($b$) of the statement ($ii$) means that the supplies $s_{ij}$ and $s_{ji}$ are interconnection neutral supply rates for the interconnections between the systems $G_i$ and $G_j$. We also emphasize that in the above theorem each matrix $P_i$ from ($i$) is indeed the same $P_i$ as in ($ii$).

The implication ($ii$) $\implies$ ($i$) is trivial to prove. It is a straightforward generalization of the Remark II.1 from two to arbitrary number of interconnected systems. The main contribution of this paper is to prove ($i$) $\implies$ ($ii$).

### B. Overview of the main results

The core part of proof of Theorem IV.2 is divided in the two subsequent sections, Section V and Section VI. Both sections contain results for which we believe are of independent interest. In this section we first formally define the problems solved in Sections V and VI. Then we indicate how these two results combine to form the proof of Theorem IV.2.

*1) Connective stability:* Consider a dynamical network given by (7) with $L \geq 2$. In Section V we will prove the following proposition.

**Proposition IV.3** *Let* $(G_i, G_j) \in E$ *and suppose that the system* (7) *admits an additive quadratic Lyapunov function of the form* (10). *Then the system obtained by interconnecting the systems* $G_i$ *and* $G_j$ *alone is stable, that is, the system given by*

$$w_{ij} = G_i v_{ij}, \quad w_{ji} = G_j v_{ji}, \quad w_{ij} = v_{ji}, \quad w_{ji} = v_{ij} \qquad (12)$$

*is stable. Moreover, the function* $V(x_i, x_j) = x_i^\top P_i x_i + x_j^\top P_j x_j$ *is a Lyapunov function for the interconnected system* (12).

We call the property from Proposition IV.3 *the connective stability*, as it is closely related to the connective stability notion defined e.g. in [9].

*2) Existence of neutral supply rates for the case of two systems:* In Section VI we will prove the following proposition.

**Proposition IV.4** *The Theorem IV.2 is true for* $L = 2$, *that is, the theorem is true for* $G = \mathrm{diag}(G_1, G_2)$ *and* $H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ *(see Figure 2).*

Note that above proposition is in fact the converse statement to the one made in Remark II.1.

*3) Proof of Theorem IV.2 (Sketch):* By Proposition IV.3, for any $(i, j)$ such that $(G_i, G_j) \in \hat{E}$ there is an additive Lyapunov function for the system (12). Then by Proposition IV.4 there exists $s_{ij}(v_{ij}, w_{ij})$ and $s_{ji}(v_{ji}, w_{ji})$ so that

a) $s_{ij}(v_{ij}, w_{ij}) + s_{ji}(v_{ji}, w_{ji}) = 0$;
b) $G_i$ is strictly dissipative with respect to $s_{ij}(v_{ij}, w_{ij})$ with the storage function $V_i(x_i) = x_i^\top P_i x_i$;
c) $G_j$ is strictly dissipative with respect to $s_{ji}(v_{ji}, w_{ji})$ with the storage function $V_i(x_i) = x_i^\top P_i x_i$.

Since this holds all pairs $(i, j)$ such that $(G_i, G_j) \in \hat{E}$, we can construct the supply rates in Theorem IV.2 so that the statement ($i$) implies the statement ($ii$). In particular, Assumption IV.1 allows us to construct such supply rates so that $G_i$ is not only strictly dissipative with respect to each $s_{ij}$ (for each $j \in N_i$) separately, but that it is also dissipative with respect to the joint combined supply $\sum_{j \in N_i} s_{ij}$, as presented in part *a)* in the statement ($ii$) of the theorem. Details of the such construction are here omitted.

## V. CONNECTIVE STABILITY

In this section we prove Proposition IV.3.

For simplicity let $i = 1$ and $j = 2$. In addition to the definitions and abbreviations made in Section III, we make the following ones. Let $\hat{v}_1$ and $\hat{w}$ be the vectors so that $v_1 = \mathrm{col}(v_{12}, \hat{v}_1)$ and $w_1 = \mathrm{col}(w_{12}, \hat{w}_1)$. Analogously, we define $\hat{v}_2$ and $\hat{w}_2$ so that $v_2 = \mathrm{col}(v_{21}, \hat{v}_2)$ and $w_2 = \mathrm{col}(w_{21}, \hat{w}_2)$. The system $G_1$ is then given in the following state-space form

$$\dot{x}_1 = A_1 x_1 + B_{12} v_{12} + \hat{B}_1 \hat{v}_1 \qquad (13a)$$
$$w_{12} = C_{12} x_1 \qquad (13b)$$
$$\hat{w}_1 = \hat{C}_1 x_1 \qquad (13c)$$

for $B_{12}$, $\hat{B}_1$, $C_{12}$ and $\hat{C}_1$ being suitably selected submatrices from $B_1$ and $C_1$. Analogously, the system $G_2$ is given by

$$\dot{x}_2 = A_2 x_2 + B_{21} v_{21} + \hat{B}_2 \hat{v}_2 \qquad (14a)$$
$$w_{21} = C_{21} x_2 \qquad (14b)$$
$$\hat{w}_2 = \hat{C}_2 x_2 \qquad (14c)$$

for $B_{21}$, $\hat{B}_2$, $C_{21}$ and $\hat{C}_2$ being suitably selected submatrices from $B_2$ and $C_2$. Since $G_1$ and $G_2$ are interconnected in a sense that $v_{21} = w_{12}$ and $v_{12} = w_{21}$, from (13) and (14) we can define the system $G_I$, given by

$$\dot{x}_I = A_I x_I + B_I v_I,$$
$$w_I = C_I x_I,$$

as interconnection of systems $G_1$ and $G_2$, where $x_I = \mathrm{col}(x_1, x_2)$, $v_I = \mathrm{col}(\hat{v}_1, \hat{v}_2)$, $w_I = \mathrm{col}(\hat{w}_1, \hat{w}_2)$, $B_I = \mathrm{diag}(\hat{B}_1, \hat{B}_2)$, $C_I = \mathrm{diag}(\hat{C}_1, \hat{C}_2)$ and

$$A_I = \begin{pmatrix} A_1 & B_{12} C_{21} \\ B_{21} C_{12} & A_2 \end{pmatrix}.$$

Now, the overall interconnected system given by (7) and presented in Figure 2 can be alternatively presented as follows

$$\begin{pmatrix} \dot{x}_I \\ \dot{x}_{II} \end{pmatrix} = \underbrace{\begin{pmatrix} A_I & 0 \\ 0 & A_{II} \end{pmatrix}}_{A_O} \begin{pmatrix} x_I \\ x_{II} \end{pmatrix} + \underbrace{\begin{pmatrix} B_I & 0 \\ 0 & B_{II} \end{pmatrix}}_{B_O} \begin{pmatrix} v_I \\ v_{II} \end{pmatrix},$$

$$\begin{pmatrix} w_I \\ w_{II} \end{pmatrix} = \underbrace{\begin{pmatrix} C_I & 0 \\ 0 & C_{II} \end{pmatrix}}_{C_O} \begin{pmatrix} x_I \\ x_{II} \end{pmatrix}, \quad \begin{pmatrix} v_I \\ v_{II} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & H_1 \\ H_2 & H_3 \end{pmatrix}}_{H_O} \begin{pmatrix} w_I \\ w_{II} \end{pmatrix},$$

where $x_{II} = \mathrm{col}(x_3, x_4, \dots, x_L)$, $v_{II} = \mathrm{col}(v_3, v_4, \dots, v_L)$, $w_{II} = \mathrm{col}(w_3, w_4, \dots, w_L)$, $A_{II} = \mathrm{diag}(A_3, A_4, \dots, A_L)$, $B_{II} = \mathrm{diag}(B_3, B_4, \dots, B_L)$, $C_{II} = \mathrm{diag}(C_3, C_4, \dots, C_L)$, and $H_1$, $H_2$ and $H_3$ are suitably constructed from the matrix $H$.

If the system (7) admits an additive Lyapunov function, with $P_I := \mathrm{diag}(P_1, P_2)$, $P_{II} := \mathrm{diag}(P_3, P_4, \dots, P_L)$, $P = \mathrm{diag}(P_I, P_{II}) \succ 0$, we have that

$$(A_O + B_O H_O C_O)^\top P + P(A_O + B_O H_O C_O) \prec 0,$$

which reads as

$$\begin{pmatrix} A_I^\top P_I + P_I A_I & M_2^\top P_{II} + P_I M_1 \\ M_1^\top P_I + P_{II} M_2 & M_3^\top P_{II} + P_{II} M_3 \end{pmatrix} \prec 0, \tag{17}$$

where $M_1 = B_I H_1 C_{II}$, $M_2 = B_{II} H_2 C_I$ and $M_3 = A_{II} + B_{II} H_3 C_{II}$. The upper left block in (17) is therefore negative definite, that is, $A_I^\top P_I + P_I A_I \prec 0$. The latter inequality and $P_I \succ 0$ imply stability of the interconnection of $G_1$ and $G_2$ alone, as stated in Proposition IV.3. This is so since such interconnected system composed of $G_1$ and $G_2$ is given by a state space realization $\dot{x}_I = A_I x_I$.

## VI. EXISTENCE OF NEUTRAL SUPPLY RATES: CASE OF TWO SYSTEMS

In this section we present a proof of Proposition IV.4.

Consider two LTI systems $G_1$ and $G_2$, given by (2) with $D_1 = 0$, $D_2 = 0$, and interconnected in a way that $v_1 = w_2 =: w$, $v_2 = w_1 =: z$, where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $z(t) \in \mathbb{R}^{n_z}$, $w(t) \in \mathbb{R}^{n_w}$. The interconnected system $G$ is then given by

$$G: \qquad \underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{} = \underbrace{\begin{pmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}. \tag{18}$$

Recall that existence of interconnection neutral supply rates means that there exist supply rates $s_1$ and $s_2$ such that $s_1(w,z) + s_2(z,w) = 0$ and

$$\dot{V}_1(x_1) < s_1(w,z), \quad \text{for} \quad \mathrm{col}(x_1, w, z) \neq 0$$
$$\dot{V}_2(x_2) < s_2(z,w) \quad \text{for} \quad \mathrm{col}(x_2, z, w) \neq 0.$$

With the abbreviations $H := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $A := \mathrm{diag}(A_1, A_2)$, $B := \mathrm{diag}(B_1, B_2)$, $C := \mathrm{diag}(C_1, C_2)$, the stability of the system (18) is by Proposition III.1 equivalent to existence of matrices $P \succ 0$ and

$$\Pi = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} := \left(\begin{array}{cc|cc} Q_{11} & Q_{12} & S_{11} & S_{12} \\ Q_{12}^\top & Q_{22} & S_{21} & S_{22} \\ \hline S_{11}^\top & S_{21}^\top & R_{11} & R_{12} \\ S_{12}^\top & S_{22}^\top & R_{12}^\top & R_{22} \end{array}\right), \tag{20}$$

which satisfy (8) and (9). Observe that, in general, both $P$ and $\Pi$ are full matrices. Proposition IV.4 is concerned with block diagonal $P$, that is, assumption is that $P = \mathrm{diag}(P_1, P_2)$. According to the full-block S-procedure, straightforward modification of Proposition III.1 states that the system (18) is stable with an additive Lyapunov function (block diagonal $P$) *if and only if* there exists full symmetric $\Pi$, i.e., as in (20), so that (8) and (9) hold. This result is the starting point of the proof. Our aim is show that with the block diagonal $P$, we can also, without loss of generality, impose certain structural constraints on $\Pi$, which in fact imply existence of the interconnection neutral supply rates. The proof follows in two steps. In the first step we make an assumption regarding rank of matrices $C_1$ and $C_2$, while in the second step we relax this assumption.

*1) Step 1:* Till *Step 2*, the standing assumption is that both $C_1$ and $C_2$ are full row rank matrices. Furthermore, we first consider the case when $n_z < n_1$ and $n_w < n_2$. At the end of *Step 1* we remark on the case when $n_z = n_1$ and $n_w = n_2$, or when we have some other combination of the above equalities/inequalities.

We make the following definitions. Let $V_1$ span the kernel of $C_1$ and $V_2$ span the kernel of $C_2$. Furthermore, let $W_1$ and $W_2$ be the matrices whose columns span the orthogonal subspaces to $V_1$ and $V_2$, respectively, and let

$$T = \begin{pmatrix} V & W & 0 \\ 0 & 0 & C_W \end{pmatrix}, \tag{21}$$

where $V = \mathrm{diag}(V_1, V_2)$, $W = \mathrm{diag}(W_1, W_2)$, $C_W = \mathrm{diag}(C_2 W_2, C_1 W_1)$. Note that $T$ is nonsingular square matrix. Let $P = \mathrm{diag}(P_1, P_2)$ and $Q, S$ and $R$ satisfy (8) and (9). After applying the congruence transformation on (9) with $T$, i.e., after pre-multiplying and post-multiplying (9) with $T^\top$ and $T$, respectively, we have

$$\begin{pmatrix} V^\top M V & V^\top M W & V^\top N C_W \\ W^\top M V & W^\top M W + W^\top C^\top R C W & W^\top N C_W + W^\top C^\top S^\top C_W \\ C_W^\top N^\top V & C_W^\top N^\top W + C_W^\top S C W & C_W^\top Q C_W \end{pmatrix} \prec 0$$

where $M := A^\top P + PA$ and $N := PB$. After applying Schur complement rule on the above inequality, with the diagonal block $V^\top M V$ to be inverted, we obtain the following equivalent inequalities

$$V^\top M V \prec 0 \tag{22a}$$

$$(\star)^\top \begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} \begin{pmatrix} CW & 0 \\ 0 & C_W \end{pmatrix} + \underbrace{(\star)^\top \begin{pmatrix} \hat{R} & \hat{S}^\top \\ \hat{S} & \hat{Q} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & C_W \end{pmatrix}}_{Y} \prec 0 \tag{22b}$$

where we have used the abbreviations

$$\hat{R} = M - MV(V^\top M V)^{-1} V^\top M,$$
$$\hat{S} = N^\top - N^\top V(V^\top M V)^{-1} V^\top M,$$
$$\hat{Q} = -N^\top V(V^\top M V)^{-1} V^\top N.$$

Note that $\hat{R}, \hat{S}$ and $\hat{Q}$ are by construction block diagonal matrices, i.e., we can write $\hat{R} = \mathrm{diag}(\hat{R}_1, \hat{R}_2)$, $\hat{S} = \mathrm{diag}(\hat{S}_1, \hat{S}_2)$, $\hat{Q} = \mathrm{diag}(\hat{Q}_1, \hat{Q}_2)$, where $\hat{R}_i \in \mathbb{R}^{n_i \times n_i}$ for $i = 1, 2$, $\hat{Q}_1 \in \mathbb{R}^{n_w \times n_w}$, $\hat{Q}_2 \in \mathbb{R}^{n_z \times n_z}$, $\hat{S}_1 \in \mathbb{R}^{n_w \times n_1}$ and $\hat{S}_2 \in \mathbb{R}^{n_z \times n_2}$. Let us define $L_1 = \begin{pmatrix} C_1 \\ V_1^\top \end{pmatrix}$, $L_2 = \begin{pmatrix} C_2 \\ V_2^\top \end{pmatrix}$. Note that $L_1 \in \mathbb{R}^{n_1 \times n_1}$ and $L_2 \in \mathbb{R}^{n_2 \times n_2}$ are nonsingular square matrices and

$$L_1 W_1 = \begin{pmatrix} C_1 W_1 \\ 0 \end{pmatrix}, \qquad L_2 W_2 = \begin{pmatrix} C_2 W_2 \\ 0 \end{pmatrix}.$$

With $L = \operatorname{diag}(L_1, L_2)$ the matrix $Y$ from (22b) can be presented as

$$Y = (\star)^\top (\star)^\top \begin{pmatrix} \hat{R} & \hat{S}^\top \\ \hat{S} & \hat{Q} \end{pmatrix} \begin{pmatrix} L^{-1}L & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & C_w \end{pmatrix},$$

or

$$Y = \begin{pmatrix} LW & 0 \\ 0 & C_W \end{pmatrix}^\top \begin{pmatrix} L^{-\top}\hat{R}L^{-1} & L^{-\top}\hat{S}^\top \\ \hat{S}L^{-1} & \hat{Q} \end{pmatrix} \begin{pmatrix} LW & 0 \\ 0 & C_W \end{pmatrix}. \tag{24}$$

Note that

$$LW = \left(\begin{array}{c|c} C_1 W_1 & 0 \\ 0 & 0 \\ \hline 0 & C_2 W_2 \\ 0 & 0 \end{array}\right) \tag{25}$$

and that we can, in conformity with the above partition of $LW$, partition $L^{-\top}\hat{R}L^{-1}$ and $\hat{S}L^{-1}$ into blocks

$$L^{-\top}\hat{R}L^{-1} = \left(\begin{array}{cc|cc} \tilde{R}^1_{11} & \tilde{R}^1_{12} & 0 & 0 \\ (\tilde{R}^1_{12})^\top & \tilde{R}^1_{22} & 0 & 0 \\ \hline 0 & 0 & \tilde{R}^2_{11} & \tilde{R}^2_{12} \\ 0 & 0 & (\tilde{R}^2_{12})^\top & \tilde{R}^2_{22} \end{array}\right), \tag{26}$$

$$\hat{S}L^{-1} = \left(\begin{array}{cc|cc} \tilde{S}^1_{11} & \tilde{S}^1_{12} & 0 & 0 \\ \hline 0 & 0 & \tilde{S}^2_{11} & \tilde{S}^2_{12} \end{array}\right). \tag{27}$$

$Y$ from (24), after multiplications and with $C_W = \operatorname{diag}(C_2 W_2, C_1 W_1)$, becomes

$$Y = \begin{pmatrix} CW & 0 \\ 0 & C_W \end{pmatrix}^\top \begin{pmatrix} \mathscr{R} & \mathscr{S}^\top \\ \mathscr{S} & \mathscr{Q} \end{pmatrix} \begin{pmatrix} CW & 0 \\ 0 & C_W \end{pmatrix} \tag{28}$$

where

$$\mathscr{R} = \begin{pmatrix} \mathscr{R}_1 & 0 \\ 0 & \mathscr{R}_2 \end{pmatrix} := \begin{pmatrix} \tilde{R}^1_{11} & 0 \\ 0 & \tilde{R}^2_{11} \end{pmatrix},$$

$$\mathscr{S} = \begin{pmatrix} \mathscr{S}_1 & 0 \\ 0 & \mathscr{S}_2 \end{pmatrix} := \begin{pmatrix} \tilde{S}^1_{11} & 0 \\ 0 & \tilde{S}^2_{11} \end{pmatrix},$$

$$\mathscr{Q} = \begin{pmatrix} \mathscr{Q}_1 & 0 \\ 0 & \mathscr{Q}_2 \end{pmatrix} := \begin{pmatrix} \hat{Q}^1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} = \hat{Q}.$$

The inequality (22b) now reads as

$$\begin{pmatrix} CW & 0 \\ 0 & C_W \end{pmatrix}^\top \left( \begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} + \begin{pmatrix} \mathscr{R} & \mathscr{S}^\top \\ \mathscr{S} & \mathscr{Q} \end{pmatrix} \right) \begin{pmatrix} CW & 0 \\ 0 & C_W \end{pmatrix} \prec 0. \tag{30}$$

Since $CW$ and $C_W$ are nonsigular square matrices, (30) is equivalent to

$$\begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} + \begin{pmatrix} \mathscr{R} & \mathscr{S}^\top \\ \mathscr{S} & \mathscr{Q} \end{pmatrix} \prec 0, \tag{31}$$

or equivalently

$$\begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} + \begin{pmatrix} \mathscr{Q} & \mathscr{S} \\ \mathscr{S}^\top & \mathscr{R} \end{pmatrix} \prec 0. \tag{32}$$

Recall that $Q$, $S$ and $R$ are full matrices, while $\mathscr{Q}$, $\mathscr{S}$ and $\mathscr{R}$ are block diagonal matrices, derived from the parameters of the systems $(A_i, B_i, C_i, i = 1, 2)$ and the Lyapunov matrices $P_1, P_2$. The derived results up to now can be summarized in the following equivalence

$$(32) \quad \Longleftrightarrow \quad (9). \tag{33}$$

After pre-multiplying (32) with $\begin{pmatrix} H \\ I \end{pmatrix}^\top$ and post-multiplying with $\begin{pmatrix} H \\ I \end{pmatrix}$, with (8), we have

$$\begin{pmatrix} H \\ I \end{pmatrix}^\top \begin{pmatrix} \mathscr{Q} & \mathscr{S} \\ \mathscr{S}^\top & \mathscr{R} \end{pmatrix} \begin{pmatrix} H \\ I \end{pmatrix} = \begin{pmatrix} \mathscr{R}_1 + \mathscr{Q}_2 & \mathscr{S}_1^\top + \mathscr{S}_2 \\ \mathscr{S}_1 + \mathscr{S}_2^\top & \mathscr{Q}_1 + \mathscr{R}_2 \end{pmatrix} \prec 0. \tag{34}$$

Now, consider the following multiplier

$$\Pi_D = \begin{pmatrix} Q_D & S_D \\ S_D^\top & R_D \end{pmatrix} \tag{35}$$

where $\alpha$ is a positive real number in the interval $(0,1)$ and

$$\begin{aligned} Q_D &= \operatorname{diag}(-\alpha\mathscr{Q}_1 + (1-\alpha)\mathscr{R}_2,\ \alpha\mathscr{R}_1 - (1-\alpha)\mathscr{Q}_2), \\ R_D &= \operatorname{diag}(-\alpha\mathscr{R}_1 + (1-\alpha)\mathscr{Q}_2,\ \alpha\mathscr{Q}_1 - (1-\alpha)\mathscr{R}_2), \\ S_D &= \operatorname{diag}(-\alpha\mathscr{S}_1 + (1-\alpha)\mathscr{S}_2^\top,\ \alpha\mathscr{S}_1^\top - (1-\alpha)\mathscr{S}_2). \end{aligned}$$

For future reference it will be convenient to use the abbreviations $Q_i^D$, $S_i^D$, $R_i^D$, $i = 1, 2$, to refer to the block diagonal matrices in $Q_D$, $S_D$, $R_D$ from (35), that is, $Q_D = \operatorname{diag}(Q_1^D, Q_2^D)$, $S_D = \operatorname{diag}(S_1^D, S_2^D)$, $R_D = \operatorname{diag}(R_1^D, R_2^D)$.

It is easy to see that with $\Pi = \Pi_D$ the condition (8) holds since we have

$$\begin{pmatrix} H \\ I \end{pmatrix}^\top \begin{pmatrix} Q_D & S_D \\ S_D^\top & R_D \end{pmatrix} \begin{pmatrix} H \\ I \end{pmatrix} = 0. \tag{36}$$

Furthermore, it also directly follows that (32) holds when $Q = Q_D$, $S = S_D$ and $R = R_D$. This is so, since with (34), by inspection it is easy to verify that

$$\begin{pmatrix} Q_D & S_D \\ S_D^\top & R_D \end{pmatrix} + \begin{pmatrix} \mathscr{Q} & \mathscr{S} \\ \mathscr{S}^\top & \mathscr{R} \end{pmatrix} \prec 0. \tag{37}$$

Now, due to (36), (37) and (33), we conclude that (8) and (9) remain satisfied when $\begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ is replaced with $\begin{pmatrix} Q_D & S_D \\ S_D^\top & R_D \end{pmatrix}$. After this replacement, it only remains to realize that then all the matrices in (9) are block diagonal and (9) decomposes into following two independent LMIs (recall $D_i = 0$)

$$\left(\begin{array}{cc} I & 0 \\ A_i & B_i \\ \hline 0 & I \\ C_i & 0 \end{array}\right)^\top \left(\begin{array}{cc|cc} 0 & P_i & 0 & 0 \\ P_i & 0 & 0 & 0 \\ \hline 0 & 0 & Q_i^D & S_i^D \\ 0 & 0 & (S_i^D)^\top & R_i^D \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_i & B_i \\ \hline 0 & I \\ C_i & 0 \end{array}\right) \prec 0, \tag{38}$$

for $i = 1, 2$, while the condition (8) reads as

$$\begin{pmatrix} Q_2^D & S_2^D \\ (S_2^D)^\top & R_2^D \end{pmatrix} = \begin{pmatrix} -R_1^D & -(S_1^D)^\top \\ -S_1^D & -Q_1^D \end{pmatrix}. \tag{39}$$

The conditions (38) specify that system $i$ is dissipative with quadratic supply $s_i$, for $i = 1, 2$, while the condition (39) implies that $s_1+s_2=0$. This concludes *Step 1* of the proof for the case when $n_z<n_1$, $n_w<n_2$. In case when either $n_z=n_1$ or $n_w=n_2$ (or both), the proof follows along the similar lines, except that the congruence transformation with the matrix $T$, defined in (21), is either completely omitted (when $n_z=n_1$ or $n_w=n_2$) or $T$ is suitably modified. We omit the details due to space limitations.

*2) Step 2: Relaxing assumption on full row rank of matrices $C_1$ and $C_2$.*

Suppose that $C_2\in\mathbb{R}^{n_w\times n_2}$ does not have full row rank, but its row rank is $\tilde{n}_w<n_w$. Without loss of generality we can write

$$C_2=\begin{pmatrix} C_2^D\\ \tilde{C}_2\end{pmatrix}$$

where $\tilde{C}_2\in\mathbb{R}^{\tilde{n}_w\times n_2}$ is a matrix collecting linearly independent rows of $C_2$, while $C_2^D$ are the remaining rows. We can write

$$C_2=\begin{pmatrix} J_2\\ I_{\tilde{n}_w}\end{pmatrix}\tilde{C}_2 \tag{40}$$

with suitably defined $J_2\in\mathbb{R}^{(n_w-\tilde{n}_w)\times\tilde{n}_w}$. Analogously, if $C_1$ is not full row rank matrix, we can define $\tilde{C}_1\in\mathbb{R}^{\tilde{n}_z\times n_1}$ as a full row rank submatrix of $C_1$ and have

$$C_1=\begin{pmatrix} J_1\\ I_{\tilde{n}_z}\end{pmatrix}\tilde{C}_1, \tag{41}$$

for some suitably defined $J_1\in\mathbb{R}^{(n_z-\tilde{n}_z)\times\tilde{n}_z}$.

Instead of considering the interconnection of systems $G_1=(A_1,B_1,C_1,0)$ and $G_2=(A_2,B_2,C_2,0)$, we can now equivalently consider stability of the system obtained by interconnecting $\tilde{G}_1=(A_1,\tilde{B}_1,\tilde{C}_1,0)$ with $\tilde{G}_2=(A_2,\tilde{B}_2,\tilde{C}_2,0)$, where

$$\tilde{B}_1=B_1\begin{pmatrix} J_2\\ I\end{pmatrix},\qquad \tilde{B}_2=B_2\begin{pmatrix} J_1\\ I\end{pmatrix}. \tag{42}$$

Since both $\tilde{C}_1$ and $\tilde{C}_2$ are full row rank, if there exists an additive Lyapunov function $V(x_1,x_2)=V_1(x_1)+V_2(x_2)=x_1^\top P_1x_1+x_2^\top P_2x_2$, we can construct a neutral supply rate (this has been proven in *Step 1*). Let $\tilde{w}$ and $\tilde{z}$ denote respectively input and output to the system $\tilde{G}_1$. Then $\tilde{z}$ and $\tilde{w}$ are the input and the output of $\tilde{G}_2$, respectively. Recall that the neutral supply rate existence implies that there exist $\tilde{Q},\tilde{R},\tilde{S}$ such that

$$\frac{d}{dt}V_1(x_1)<\begin{pmatrix}\tilde{w}\\ \tilde{z}\end{pmatrix}^\top\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^\top & \tilde{R}\end{pmatrix}\begin{pmatrix}\tilde{w}\\ \tilde{z}\end{pmatrix},$$

for all $\mathrm{col}(x_1,\tilde{w},\tilde{z})\neq 0$ as trajectories of system $\tilde{G}_1$;

$$\frac{d}{dt}V_2(x_2)<-\begin{pmatrix}\tilde{w}\\ \tilde{z}\end{pmatrix}^\top\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^\top & \tilde{R}\end{pmatrix}\begin{pmatrix}\tilde{w}\\ \tilde{z}\end{pmatrix},$$

for all $\mathrm{col}(x_2,\tilde{z},\tilde{w})\neq 0$ as trajectories of system $\tilde{G}_2$.

The above dissipation inequalities are equivalent to the following matrix inequalities

$$(\star)^\top\begin{pmatrix}0 & P_1\\ P_1 & 0\end{pmatrix}\begin{pmatrix}I & 0\\ A_1 & \tilde{B}_1\end{pmatrix}\prec(\star)^\top\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^\top & \tilde{R}\end{pmatrix}\begin{pmatrix}0 & I\\ \tilde{C}_1 & 0\end{pmatrix}, \tag{44}$$

$$(\star)^\top\begin{pmatrix}0 & P_2\\ P_2 & 0\end{pmatrix}\begin{pmatrix}I & 0\\ A_2 & \tilde{B}_2\end{pmatrix}\prec(\star)^\top\begin{pmatrix}-\tilde{R} & -\tilde{S}^\top\\ -\tilde{S} & -\tilde{Q}\end{pmatrix}\begin{pmatrix}0 & I\\ \tilde{C}_2 & 0\end{pmatrix}. \tag{45}$$

Consider the equality

$$(\star)^\top\left(\begin{array}{cc|cc} Q_{11} & Q_{12} & S_{11} & S_{12}\\ Q_{12}^\top & Q_{22} & S_{21} & S_{22}\\ \hline S_{11}^\top & S_{21}^\top & R_{11} & R_{12}\\ S_{12}^\top & S_{22}^\top & R_{12}^\top & R_{22}\end{array}\right)\left(\begin{array}{c|c} J_2 & 0\\ I & 0\\ \hline 0 & J_1\\ 0 & I\end{array}\right)=\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^\top & \tilde{R}\end{pmatrix}, \tag{46}$$

which is a linear equation in $Q,S,R$, where

$$Q:=\begin{pmatrix}Q_{11} & Q_{12}\\ Q_{12}^\top & Q_{22}\end{pmatrix},\ S:=\begin{pmatrix}S_{11} & Q_{12}\\ S_{21} & S_{22}\end{pmatrix},\ R:=\begin{pmatrix}R_{11} & R_{12}\\ R_{12}^\top & R_{22}\end{pmatrix},$$

for some known $\tilde{Q},\tilde{S},\tilde{R}$.

Substituting (46) into (44), with (41) and (42), we obtain

$$\begin{aligned}&(\star)^\top\underbrace{\begin{pmatrix}I & 0\\ A_1 & B_1\end{pmatrix}^\top\begin{pmatrix}0 & P_1\\ P_1 & 0\end{pmatrix}\begin{pmatrix}I & 0\\ A_1 & B_1\end{pmatrix}}_{=:X_1}\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_2\\ 0 & I\end{array}\right)\prec\\ &\prec(\star)^\top\underbrace{\begin{pmatrix}0 & I\\ C_1 & 0\end{pmatrix}^\top\begin{pmatrix}Q & S\\ S^\top & R\end{pmatrix}\begin{pmatrix}0 & I\\ C_1 & 0\end{pmatrix}}_{=:Y_1}\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_2\\ 0 & I\end{array}\right).\end{aligned} \tag{47}$$

Similarly, substituting (46) into (45), with (40) and (42), we obtain

$$\begin{aligned}&\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_1\\ 0 & I\end{array}\right)^\top\underbrace{\begin{pmatrix}I & 0\\ A_2 & B_2\end{pmatrix}^\top\begin{pmatrix}0 & P_2\\ P_2 & 0\end{pmatrix}\begin{pmatrix}I & 0\\ A_2 & B_2\end{pmatrix}}_{=:X_2}\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_1\\ 0 & I\end{array}\right)\prec\\ &\prec\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_1\\ 0 & I\end{array}\right)^\top\underbrace{\begin{pmatrix}0 & I\\ C_2 & 0\end{pmatrix}^\top\begin{pmatrix}-R & -S^\top\\ -S & -Q\end{pmatrix}\begin{pmatrix}0 & I\\ C_2 & 0\end{pmatrix}}_{=:Y_2}\left(\begin{array}{c|c} I & 0\\ \hline 0 & J_1\\ 0 & I\end{array}\right).\end{aligned} \tag{48}$$

Our aim is to show that we can always select $Q,S$ and $R$ with (46) such that $X_1\prec Y_1$ and $X_2\prec Y_2$. Indeed, these two inequalities mean that $Q,S$ and $R$ define the interconnection neutral supply rates for the original system with matrices $C_1$ and $C_2$.

Consider first the inequality $X_1\prec Y_1$. The inequality (47) implies that $X_1\prec Y_1$ on $\mathrm{Im}\begin{pmatrix}I & 0\\ 0 & J_2\\ 0 & I\end{pmatrix}$, that is, $x^\top X_1x<x^\top Y_1x$ for all $x\in\mathrm{Im}\begin{pmatrix}I & 0\\ 0 & J_2\\ 0 & I\end{pmatrix}$, $x\neq 0$, but not necessarily also for an arbitrary $x\neq 0$. Note that $\mathrm{Ker}\begin{pmatrix}I & 0 & 0\\ 0 & J_2^\top & I\end{pmatrix}=\mathrm{Im}\begin{pmatrix}0\\ I\\ -J_2^\top\end{pmatrix}$, and

$$K:=\left(\begin{array}{c|cc} I & 0 & 0\\ \hline 0 & J_2 & I\\ 0 & I & -J_2^\top\end{array}\right)$$

is nonsingular square matrix. The dashed lines in the above definition of $K$ indicate partition into $2\times 2$ matrix blocks whose dimensions are in conformity with matrix blocks in $X_1$ and $Y_1$, allowing for direct block-wise multiplications in expressions $K^\top X_1K$ and $K^\top Y_1K$.

Next, we show that we can always select $Q,S$ and $R$ in (46) so that $K^\top X_1K\prec K^\top Y_1K$. Since $K$ is nonsingular square matrix, the latter inequality indeed implies $X_1\prec Y_1$

In addition to (46) let us further constrain $Q$ by adding the following relation between $Q$ and $\tilde{Q}$

$$\begin{pmatrix} J_2 & I \\ I & -J_2^\top \end{pmatrix}^\top \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{pmatrix} \begin{pmatrix} J_2 & I \\ I & -J_2^\top \end{pmatrix} = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & \gamma_Q I \end{pmatrix} \tag{49}$$

for some fixed real $\gamma_Q$. Note that for given $\tilde{Q}$ and $\gamma_Q$, the above equation uniquely defines $Q$. Also note that the only constraint on $Q$ from (46) is given by

$$\begin{pmatrix} J_2 \\ I \end{pmatrix}^\top Q \begin{pmatrix} J_2 \\ I \end{pmatrix} = \tilde{Q}$$

and is also present in (49). In that sense, uniquely defined $Q$ from (49) necessarily satisfies constraint on $Q$ imposed by (46).

The inequality (47), after multiplications and substitution for $\tilde{Q}$, reads as

$$\begin{pmatrix} A_1^\top P_1 + P_1 A_1 - C_1^\top R C_1 & \left(P_1 B_1 - C_1^\top S^\top\right)\begin{pmatrix} J_2 \\ I \end{pmatrix} \\ \begin{pmatrix} J_2^\top & I \end{pmatrix}\left(B_1^\top P_1 - S C_1\right) & -\tilde{Q} \end{pmatrix} \prec 0, \tag{50}$$

while $K^\top X_1 K \prec K^\top Y_1 K$, after multiplications and with (49), reads as

$$\begin{pmatrix} A_1^\top P_1 + P_1 A_1 - C_1^\top R C_1 & \left(P_1 B_1 - C_1^\top S^\top\right)\begin{pmatrix} J_2 \\ I \end{pmatrix} & N \\ \begin{pmatrix} J_2^\top & I \end{pmatrix}\left(B_1^\top P_1 - S C_1\right) & -\tilde{Q} & 0 \\ N^\top & 0 & -\gamma_Q I \end{pmatrix} \prec 0, \tag{51}$$

where $N = \left(P_1 B_1 - C_1^\top S^\top\right)\begin{pmatrix} I & -J_2 \end{pmatrix}^\top$. Applying the Schur complement rule on the above inequality, with lower right block $(-\gamma_Q I)$ inverted, we obtain that (51) is equivalent to

$$\begin{pmatrix} A_1^\top P_1 + P_1 A_1 - C_1^\top R C_1 + \gamma_Q^{-1} N N^\top & \left(P_1 B_1 - C_1^\top S^\top\right)\begin{pmatrix} J_2 \\ I \end{pmatrix} \\ \begin{pmatrix} J_2^\top & I \end{pmatrix}\left(B_1^\top P_1 - S C_1\right) & -\tilde{Q} \end{pmatrix} \prec 0, \tag{52}$$

with $\gamma_Q > 0$. Due to (50), which is guaranteed to hold, we can always render (52) feasible by taking sufficiently large $\gamma_Q$. To summarize, with sufficiently large $\gamma_Q$, the equation (49) gives us the parameter matrix $Q$ for neutral supply rate of the original system, starting from the parameter matrix $\tilde{Q}$ of the modified system.

Satisfying the inequality $X_2 \prec Y_2$ follows by symmetry and as a result gives us the following conditions which relate $R$ with $\tilde{R}$:

$$\begin{pmatrix} J_1 & I \\ I & -J_1^\top \end{pmatrix}^\top \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^\top & R_{22} \end{pmatrix} \begin{pmatrix} J_1 & I \\ I & -J_1^\top \end{pmatrix} = \begin{pmatrix} \tilde{R} & 0 \\ 0 & \gamma_R I \end{pmatrix} \tag{53}$$

for some sufficiently small negative real $\gamma_R$ (sufficiently large $|\gamma_R|$) Due to space limitations we will not present the detailed proof of why (53) satisfies. The procedure is completely analogous to the one for $Q$.

Finally, to complete the proof, note that conditions $K^\top X_1 K \prec K^\top Y_1 K$ and $F^\top X_2 F \prec F^\top Y_2 F$, with (49) and (53), do not impose any additional constraints on $S$, that is, the only constraints on $S$ that we consider are the ones imposed by (46), and it is easy to see that they always have a solution. More precisely, (46) gives the following relation between $S$ and $\tilde{S}$

$$\begin{pmatrix} J_2^\top & I \end{pmatrix} \underbrace{\begin{pmatrix} S_{12} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}}_{S} \begin{pmatrix} J_1 \\ I \end{pmatrix} = \tilde{S},$$

which always has a solution in $S$ for any given $\tilde{S}$.

## VII. CONCLUSIONS

In this paper we have proved that existence of additive quadratic Lyapunov function for an acyclic LTI dynamical network implies existence of suitably defined interconnection neutral supply rates.

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